Selected Solutions, Leon §6.1

6.1.1 Find the eigenvalues and associated eigenspaces of each of the following matrices.

(e) $A = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$. The characteristic polynomial is $p(\lambda) = \lambda^2 - 4\lambda + 5$, with roots $\lambda_1 = 2 - i$ and $= \lambda_2 = 2 + i$. We know that the associated eigenvectors will come in conjugate pairs, so our work is cut in half. We find that

$$A - \lambda_1 I = \begin{bmatrix} -1+i & 1 \\ -2 & 1+i \end{bmatrix},$$

and verify that the matrix is singular (row 2 is $(1+i)$ times row 1). If the eigenvector we seek has the form $(x_1, x_2)^T$, then setting $x_2 = s$ we have $2x_1 = (1+i)s$, or $x_1 = \left(\frac{1+i}{2}\right) s$. If we choose $s = 2$, we have $\mathbf{x} = (1+i, 2)^T$. The associated eigenspace is Span($\mathbf{x}$). The eigenspace associated with λ_2, then, is Span $\left((1-i, 2)^T\right)$.

(f) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. The characteristic polynomial is $p(\lambda) = -\lambda^3$. Setting $p(\lambda) = 0$, we find that $\lambda = 0$ is an eigenvalue of algebraic multiplicity 3. But $N(A - 0I) = N(A) = \text{Span}\left((1,0,0)^T\right)$, a 1-dimensional subspace of R^3.

(g) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. The characteristic polynomial is $p(\lambda) = (1-\lambda)^2(2-\lambda)$. Setting $p(\lambda) = 0$, we find $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$. It remains to be seen whether we can find two linearly independent eigenvectors associated with the repeated eigenvalue. We compute

$$A - I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The free variables are x_1 and x_3, and an eigenvector has the form

$$\mathbf{x} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

and it follows that a basis for the eigenspace associated with $\lambda_1 = \lambda_2 = 1$ is $\{(1,0,0)^T, (0,1,-1)^T\}$. Moving on to the eigenspace associated with $\lambda_3 = 2$, we find

$$A - 2I = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix},$$

from which we see that $x_3 = 0$ and $x_1 = x_2$. We might choose $\mathbf{x} = (1, 1, 0)$ for a basis for the associated eigenspace.

6.1.3 Let A be an $n \times n$ matrix. Prove that A is singular if and only if $\lambda = 0$ is an eigenvalue of A.

Proof: Let A be an $n \times n$ matrix. Suppose that A is singular. Then A has a nontrivial nullspace, so we can choose $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0} = 0\mathbf{x}$, from which we see that $\mathbf{x}$ is an eigenvalue for $\lambda = 0$. The converse follows by a symmetric argument. □

6.1.4 Let A be a nonsingular matrix and let λ be an eigenvalue of A. Show that $1/\lambda$ is an eigenvalue of A^{-1}.

Proof: Let A and λ be as described, and suppose that $\mathbf{x}$ is an eigenvector associated with λ, which is nonzero (see the preceding exercise). Then

$$\begin{aligned}
\mathbf{x} &= (A^{-1}A)\mathbf{x} \\
&= A^{-1}(A\mathbf{x}) \\
&= A^{-1}\lambda\mathbf{x} \\
&= \lambda A^{-1}\mathbf{x},
\end{aligned}$$

and it follows that $A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$. □

6.1.5 Let λ be an eigenvalue of A with associated eigenvector $\mathbf{x}$. Prove by induction that $\mathbf{x}$ is an eigenvector of A^m, associated with the eigenvalue λ^m, for each $m \geq 1$.

Proof: Let A, λ, and $\mathbf{x}$ be as described. The result is obvious when $m = 1$. So assume that $A^k\mathbf{x} = \lambda^k\mathbf{x}$ for some $k \geq 1$. Then

$$\begin{aligned}
A^{m+1}\mathbf{x} &= (AA^k)\mathbf{x} \\
&= A(A^k\mathbf{x}) \\
&= A(\lambda^k\mathbf{x}) \\
&= \lambda^k(A\mathbf{x}) \\
&= \lambda^k(\lambda\mathbf{x}) \\
&= \lambda^{k+1}\mathbf{x},
\end{aligned}$$

and we're done. □

6.1.6 Let $A \in \mathbf{R}^{n\times n}$ be an idempotent matrix, i.e., $A^2 = A$. Show that if λ is an eigenvalue of A, then $\lambda \in \{0, 1\}$.

Proof: Let λ be an eigenvalue of A, with associated eigenvector $\mathbf{x}$. By the preceding result, λ^2 is an eigenvalue of A^2, with associated eigenvector $\mathbf{x}$. Since A is idempotent, then

$$\lambda^2\mathbf{x} = A^2\mathbf{x} = A\mathbf{x} = \lambda\mathbf{x},$$

but then $\lambda^2 = \lambda$, and it follows that either $\lambda = 1$ or $\lambda = 0$. $\square$

6.1.7 Suppose $A \in \mathbf{R}^{n\times n}$ is *nilpotent*, i.e., $A^k = \mathbf{0}$ for some $k \in \mathbf{Z}^+$. Show that A has no nonzero eigenvalue.

Proof: Let A be as described, and suppose $A\mathbf{x} = \lambda\mathbf{x}$. Then $\lambda^k\mathbf{x} = A^k\mathbf{x} = \mathbf{0}$. Since $\mathbf{x} \neq 0$, it must be that $\lambda = 0$. $\square$

6.1.8 Let $A \in \mathbf{R}^{n\times n}$, and let $B = A = \alpha I$ for some scalar α. How do the eigenvalues of A and B compare?

Solution: Let A and B be as described. Suppose $A\mathbf{x} = \lambda\mathbf{x}$. Then

$$\begin{aligned} B\mathbf{x} &= (A - \alpha I)\mathbf{x} \\ &= A\mathbf{x} - \alpha\mathbf{x} \\ &= \lambda\mathbf{x} - \alpha\mathbf{x} \\ &= (\lambda - \alpha)\mathbf{x}. \end{aligned}$$

So if λ is an eigenvalue of A, with associated eigenvector $\mathbf{x}$, then $\lambda - \alpha$ is an eigenvector of $B = A - \alpha I$, also with associated eigenvector $\mathbf{x}$.

6.1.9 Let A be an $n \times n$ matrix. Show that A and A^T have the same eigenvalues. Do they also share eigenvectors?

Solution: That A and A^T have identical eigenvalues follows from the fact that the determinant of a matrix is the determinant of its transpose. It turns out that A and A^T do not necessarily have the same eigenvectors, though. For example, let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

This matrix has a repeated eigenvalue $\lambda = 0$, but does not possess a full set of linearly independent eigenvectors. An eigenvector for A has the form $\mathbf{x} = (s, 0)^T$, while an eigenvector for A^T has the form $\mathbf{x} = (0, s)^T$.

6.1.11 Let A be a 2×2 matrix satisfying $\mathrm{tr}(A) = 8$ and $\det(A) = 12$. What are the eigenvalues of A?

Solution: Let λ_1, λ_2 denote the eigenvalues of A. By hypothesis, $\lambda_1 + \lambda_2 = 8$ and $\lambda_1\lambda_2 = 12$. Substituting $\lambda_2 = 8 - \lambda_1$ in the second equation leads to the quadratic equation, $\lambda_1^2 - 8\lambda_1 + 12 = 0$. The solutions are $\lambda_1 = 2$ and $\lambda_1 = 6$. If we let $\lambda_1 = 2$, then $\lambda_2 = 8 - 2 = 6$; if we let $\lambda_1 = 6$, then $\lambda_2 = 8 - 6 = 2$.

6.1.15 Let A be an $n \times n$ matrix, and let λ be an eigenvalue of A. If $A - \lambda I$ has rank k, what is the dimension of the eigenspace corresponding to λ?

Solution: By Theorem 3.6.4, the sum of the rank of $A - \lambda I$ and the nullity of $A - \lambda I$ is equal to n. Since the eigenspace corresponding to λ is precisely the nullspace of $A - \lambda I$, we see that the dimension of that eigenspace is $n - k$.

6.1.19 Let $B = S^{-1}AS$, and let $\mathbf{x}$ be an eigenvector of B belonging to λ. Show that $S\mathbf{x}$ is an eigenvector of A belonging to λ.

Proof: Let A, B, λ, and $\mathbf{x}$ be as described. Then

$$\begin{aligned} A(S\mathbf{x}) &= AS\mathbf{x} \\ &= (SS^{-1})AS\mathbf{x} \\ &= S(S^{-1}AS\mathbf{x}) \\ &= S(B\mathbf{x}) \\ &= S(\lambda\mathbf{x}) \\ &= \lambda(S\mathbf{x}), \end{aligned}$$

and the result follows. $\square$